Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at http://about.jstor.org/participate-jstor/individuals/early-journal-content.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.

ON THE EXPANSION OF CERTAIN ANALYTIC FUNCTIONS IN SERIES.*

By R. D. Carmichael.

Introduction. Let ρ be a positive constant and let $g(x)$ be a function of x with the asymptotic representation

$$g(x) \sim x^{\mu-\rho x}e^{\alpha+\beta x}\left(1+\frac{a_1}{x}+\frac{a_2}{x^2}+\cdots\right) \tag{1}$$

valid in a sector V including in its interior the positive axis of reals and every point of the imaginary axis. Moreover let $g(x)$ be analytic at every point x in V for which $|x|$ is greater than a given constant γ. By means of this function $g(x)$ we set up the class of series which have the form

$$S(x) = \sum_{k=0}^{\infty} c_k \frac{g(x+k)}{g(x)}, \tag{2}$$

where $c_0, c_1, c_2, \cdots$ are constants. These (and other more general) series I have already treated in several papers.† The object of this note is to derive certain necessary and sufficient conditions for the representation of functions in the form of series $S(x+a)$ where a is a suitable constant depending on the function to be represented.

1. **Representation of a function of a certain class by means of a suitable integral along a straight line.** Let $f(x)$ be a function of x which is analytic at every finite point to the right of a line L_1 parallel to the axis of imaginaries. Let L denote a line parallel to L_1 and to its right; and let us think of L as having the direction which leads from a point with negative imaginary part to one with positive imaginary part. On L let $f(x)$ be bounded by the inequality

$$|f(x)| < \frac{N}{|x|^{\epsilon}}, \tag{3}$$

when $|x|$ is greater than a sufficiently large positive constant X, where ϵ and N are given positive constants. We shall show that *such a function*

---

* Presented to the American Mathematical Society, April, 1920.

† These memoirs will be referred to by the numbers in the following list: I. Trans. Am. Math. Soc., vol. 17 (1916): 207–232. II. Bulletin Am. Math. Soc., vol. 23 (1917): 407–425. III. American Journal of Mathematics, vol. 39 (1917): 385–403. IV. American Journal of Mathematics, vol. 40 (1918): 113–126.

$f(x)$ may be written in the form of the integral

$$f(x) = \frac{1}{2\pi i}\int_L \frac{f(z)dz}{x - z}, \tag{4}$$

where the integral is taken along L in the direction already indicated and is valid for every point x to the right of L. For this purpose consider the semicircle of radius t about a point α on L as center and lying to the right of L. Let Δ_t denote the region inclosed by the semicircle C_t and its diameter D_t. Let x be any point to the right of L and let t be so large that x is in the interior of Δ_t. Then by the Cauchy integral formula we have

$$f(x) = \frac{1}{2\pi i}\int_{D_t} \frac{f(z)dz}{z - x} + \frac{1}{2\pi i}\int_{C_t} \frac{f(z)dz}{z - x},$$

where the integrals are taken along D_t and C_t in a positive sense with reference to the inclosed area Δ_t.

Now let the radius t of C_t become infinite. The second integral in the last equation approaches zero, as one sees readily by taking the absolute value of the integrand and employing inequality (3). The first integral in the last equation approaches the integral in (4). Hence $f(x)$ can be written in the form given in (4) and the representation is valid for every point x to the right of L.

2. **The convergence of certain asymptotic representations with respect to** $g(x)$. Let $f(x)$ be a function which is analytic at every (finite) point to the right of a line L_1 parallel to the axes of imaginaries. Denote by L a line to the right of L_1 and parallel to it. Let a constant a be chosen so that for x on L the real part $R(x + a)$ of $x + a$ is greater than a non-negative integer r and so that $g(x + a + 1)/g(x + a)$ is analytic and different from zero at every finite point to the right of the line L_1. Then for every positive integer n the function $g(x + a + n)/g(x + a)$ has the last-mentioned property since

$$\frac{g(x + a + n)}{g(x + a)} = \frac{g(x + a + 1)}{g(x + a)} \cdot \frac{g(x + a + 2)}{g(x + a + 1)} \cdots \frac{g(x + a + n)}{g(x + a + n - 1)}. \tag{5}$$

Let $f(x)$ be written in the form

$$f(x) = \sum_{k=0}^{n} c_k \frac{g(x + a + k)}{g(x + a)} + R_n(x)\frac{g(x + a + n + 1)}{g(x + a)}. \tag{6}$$

For every x on L and for every positive integer n let $R_n(x)$ satisfy the inequality

$$|R_n(x)| \leqq (n!)^{\rho} e^{nR(\rho-\beta)} n^{r\rho + R(\frac{1}{2}\rho - n)} \epsilon_n, \tag{7}$$

where ϵ_n approaches zero with $1/n$. Moreover, for x to the right of L, let $f(x)$ be such that the function $f(x) - c_0$ is bounded by the inequality

$$|f(x) - c_0| < N|x|^{-\epsilon}$$

when $|x|$ is greater than a sufficiently large positive quantity X, where ϵ and N are positive constants. Then we shall show that $f(x)$ *may be represented by the series*

$$f(x) = \sum_{k=0}^{\infty} c_k \frac{g(x+a+k)}{g(x+a)} \tag{8}$$

and that this series converges for every x to the right of L.

From the asymptotic character of $g(x)$ it follows readily that $(x+a)^{s\rho}g(x+a+s)/g(x+a)$, and hence $x^{s\rho}g(x+a+s)/g(x+a)$, approaches a finite non-zero limit as $x+a$ approaches infinity in V. Hence when x is to the right of L it is easy to see that the remainder term in (6) may be represented in the form of an integral in accordance with the result in § 1. Hence we have

$$f(x) = \sum_{k=0}^{n} c_k \frac{g(x+a+k)}{g(x+a)} + \frac{1}{2\pi i}\int_L \frac{R_n(z)}{x-z}\,\frac{g(z+a+n+1)}{g(z+a)}\,dz, \tag{9}$$

where x is to the right of L and the integration along L is taken in the direction indicated in § 1 and x is to the right of L.

In order to prove that $f(x)$ has the convergent representation (7) it is now necessary and sufficient to show that the integral in the last equation approaches the value zero as n becomes infinite. But this integral is not greater in absolute value than the integral

$$\int_L \left|\frac{R_n(z)}{z-x}\right| \cdot \left|\frac{g(z+a+n+1)}{g(z+a)}\right| \cdot |dz|.$$

In order to find a function of n which dominates the latter integral we employ the asymptotic relation

$$\begin{aligned} g(x)\{\Gamma(x)\}^{\rho} &\sim x^{\mu-\rho x}e^{\alpha+\beta x}\left(1+\frac{a_1}{x}+\cdots\right)\cdot x^{-\frac{1}{2}+x}e^{-x}\sqrt{2\pi}\left(1+\frac{b_1}{x}+\cdots\right)^{\rho} \\ &\sim x^{\mu-\rho/2}e^{\alpha+(\beta-\rho)x}(\sqrt{2\pi})^{\rho}\left(1+\frac{c_1}{x}+\cdots\right). \end{aligned}$$

From this we have

$$\frac{g(x+n+1)}{g(x)}\left\{\frac{\Gamma(x+n+1)}{\Gamma(x)}\right\}^{\rho} = \left(1+\frac{n}{x}\right)^{\mu-\rho/2} e^{(\beta-\rho)n}S_{(x,\,n)},$$

where $S(x, n)$ approaches 1 when n becomes infinite or x becomes infinite in V or both of these conditions are realized simultaneously. Hence for z on L constants M_1 and M_2 exist such that

$$(10)\quad \left|\frac{g(z+a+n+1)}{g(z+a)}\right|$$

$$< M_1|(z+a)(z+a+1)\cdots(z+a+n)|^{-\rho}e^{nR(\beta-\rho)}n^{R(\mu-\frac{1}{2}\rho}$$

$$< M_2|z+a|^{-\rho}(n!)^{-\rho}e^{nR(\beta-\rho)}n^{R(\mu-\frac{1}{2}\rho)-r\rho}.$$

Employing this inequality and (7) we see that a constant M exists such that the last-written integral has a value less than $M\epsilon_n$. Hence it approaches zero with $1/n$. Hence from (9) we have (8), a relation which is valid if x is to the right of L, as was to be proved.

Now let L_2 be a line to the right of L and parallel to it. Since $g(x+a+n)/g(x+a)$ is analytic and different from zero at every point to the right of L_1, it follows that there is no exceptional point for the series in (8) and no limit point of such points to the right of L. Therefore, from theorem III of memoir III it follows that the series in (8) is uniformly convergent for x in the region consisting of L_2 and the half-plane to its right (exclusive of the point infinity).

We are thus led to the following theorem:

Let $f(x)$ be a function which is analytic at every (finite) point to the right of a line L_1 parallel to the axis of imaginaries and suppose that constants c_0 and $\epsilon(\epsilon > 0)$ exist such that $|x|^{\epsilon}\cdot|f(x)-c_0|$ is bounded as x approaches infinity to the right of L_1. Denote by L a line to the right of L_1 and parallel to it. Let a constant a be chosen so that for X on L $R(x+a)$ is greater than a non-negative integer r and so that $g(x+a+1)/g(x+a)$ is analytic and different from zero at every finite point to the right of L_1. If $f(x)$ is written in the form (6) then let $R_n(x)$, for every x on L and for every positive integer n, satisfy inequality (7). Then $f(x)$ may be represented by the series in (8) and this series converges for every finite x to the right of L. Moreover, it converges uniformly in the region consisting of a line to the right of and parallel to L and the half-plane to the right of such line (exclusive of the point infinity).

3. **Necessary and sufficient conditions for the representation of a function $f(x)$ in the form of a series $S(x+a)$.** Let $F(x)$ be a function represented in the form of a series $S(x)$,

$$(11)\qquad F(x) = \sum_{k=0}^{\infty}\gamma_k\frac{g(x+k)}{g(x)},$$

converging for some value of x which is non-exceptional for this series and hence in a half-plane $R(x) > l$, where the line $R(x) = l$ is the boundary of the region of convergence.

From theorem I of memoir IV it follows that a constant S_1 exists such that

$$|\gamma_n g(n)| < n^{s_1}, \quad n \geqq 2.$$

Now we have with respect to n the asymptotic formula

$$g(n)\{\Gamma(n)\}^\rho \sim e^{n(\beta-\rho)} n^{\mu-\frac{1}{2}\rho} e^{\alpha}(\sqrt{2\pi})^\rho \left(1 + \frac{c_1}{n} + \cdots\right).$$

Hence a constant s exists such that

$$|\gamma_{n+1}| < (n!)^\rho e^{nR(\rho-\beta)} n^s, \quad n \geqq 2. \tag{12}$$

Let us now write $F(x)$ in the form

$$F(x) = \sum_{k=0}^{n} \gamma_k \frac{g(x+k)}{g(x)} + \bar{R}_n(x) \frac{g(x+n+1)}{g(x)}.$$

Then we have

$$\bar{R}_n(x) = \sum_{k=n+1}^{\infty} \gamma_k \frac{g(x+k)}{g(x+n+1)} \tag{13}$$

provided that x is confined to a portion of the region of convergence of the series in (11) for which $g(x+n+1)/g(x)$ is different from zero. If x is on a line L parallel to the axis of imaginaries and sufficiently far to its right, we may employ relation (10) with z replaced by x, a by $n+1$, and n by $k-n-2$, to conclude to the existence of a constant M_3 (independent of x and n) such that

$$\left|\frac{g(x+k)}{g(x+n-1)}\right| < M_3 |(x+n+1)\cdots(x+k-1)|^{-\rho} e^{(k-n-2)R(\beta-\rho)}(k-n-2)^{R(\mu-\frac{1}{2}\rho)},$$

where $k > n+2$. Hence we have

$$\begin{aligned}
&\left|\gamma_k \frac{g(x+k)}{g(x+n+1)}\right. \\
&\qquad < M_3 \left\{\frac{(k-1)!}{(x+n-1)\cdots(x+k-1)}\right\}^\rho e^{(n+1)R(\rho-\beta)}(k-n-2)^{R(\mu-\frac{1}{2}\rho)} n^s \\
&\qquad < M_3 e^{R(\rho-\beta)}(n!)^\rho e^{nR(\rho-\beta)} n^s \left\{\frac{(n+1)\cdots(k-1)}{(x+n+1)\cdots(x+k-1)}\right\}^\rho (k-n-2)^{R(\mu-\frac{1}{2}\rho)} \\
&\qquad < M_4 (n!)^\rho e^{nR(\rho-\beta)} n^{s+t\rho} \left\{\frac{(n+t+1)\cdots(k-1)}{(x+n+1)\cdots(x+n+k-t-1)}\right\}^\rho \\
&\qquad\qquad \cdot k^{-\rho t}(k-n-2)^{R(\mu-\frac{1}{2}\rho)},
\end{aligned}$$

where t is a positive integer such that $\rho t > R(\mu - \frac{1}{2}\rho) + 2$, $k > n + t + 2$, and M_4 is a suitable constant. Hence if the line L is sufficiently far to the right, then for x on L we have

$$\left|\gamma_k \frac{g(x+k)}{g(x+n+1)}\right| < M_5(n!)^\rho e^{nR(\rho-\beta)} n^{s+t\rho} k^{-2}, \qquad k > n+t+2,$$

where M_5 is a suitable quantity independent of x and n.

Employing the last inequality and relations (12) and (13) we find that for x on L we have

$$|\bar{R}_n(x)| < \sum_{k=n+1}^{n+t+2} [(k-1)!]^\rho e^{(k-1)R(\rho-\beta)} (k-1)^s \left|\frac{g(x+k)}{g(x+n+1)}\right| + \sum_{k=n+t+3}^{\infty} M_5(n!)^\rho e^{nR(\rho-\beta)} n^{s+t\rho} \frac{1}{k^2}.$$

Hence a quantity M_6, independent of x and n, exists such that for x on L we have

$$|\bar{R}_n(x)| < M_6(n!)^\rho e^{nR(\rho-\beta)} n^{s+t\rho}.$$

Therefore an r exists such that $\bar{R}_n(x)$ satisfies the condition imposed on $R_n(x)$ in inequality (7), provided that the line L has been taken sufficiently far to the right.

It may now be readily shown that for the given function $F(x)$ lines L_1 and L exist such that $F(x)$ has the same properties with respect to them and to a constant a (in this case zero) as are specified for $f(x)$ in the hypothesis of the theorem of § 2. Analyticity to the right of some line L_1 parallel to the axis of imaginaries follows from the last result in § 3 of memoir I. From theorem I of memoir III it follows that $|x|^\rho \cdot |F(x) - \gamma_0|$ is bounded for x approaching infinity in a half-plane to the right of a suitable line L_1. We have just demonstrated the appropriate property of the factor $\bar{R}_n(x)$ in the remainder term. Hence the conditions given in the theorem of § 2 and there found to be sufficient for the representation of a function $f(x)$ in the form of a series $S(x+a)$ are also properties of every function so represented. *Thus we have necessary and sufficient conditions for the representation of functions in the form of series $S(x+a)$.*

UNIVERSITY OF ILLINOIS.